

The unconditional case of the complex S -inequality

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Abstract

In this note we prove the complex counterpart of the S -inequality for complete Reinhardt sets. In particular, this result implies that the complex S -inequality holds for unconditional convex sets.

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1 Introduction

Studying various aspects of a Gaussian measure in a Banach space one often needs precise estimates on measures of balls and their dilations. This gives rise to the question how the function $(0, \infty) \ni t \mapsto \mu(tB)$ behaves. Here B is a convex and symmetric subset of some Banach space, i.e. a unit ball with respect to some norm, and μ is a Gaussian measure. Thanks to certain approximation arguments we may only deal with the simplest spaces, namely \mathbb{R}^n or \mathbb{C}^n . In the former case the issue is well understood due to R. Latała and K. Oleszkiewicz. Denote by γ_n the standard Gaussian measure on \mathbb{R}^n , i.e. the measure with the density at a point (x_1, \dots, x_n) equal to $\frac{1}{\sqrt{2\pi}^n} \exp(-x_1^2/2 - \dots - x_n^2/2)$. In [LO1] it is shown that for a symmetric convex body $K \subset \mathbb{R}^n$ and the strip $P = \{x \in \mathbb{R}^n \mid |x_1| \leq p\}$, where p is chosen so that $\gamma_n(K) = \gamma_n(P)$, we have

$$\gamma_n(tK) \geq \gamma_n(tP), \quad t \geq 1.$$

This result is called *S -inequality*. The interested reader is also referred to a concise survey [Lat].

In the present note we would like to focus on S -inequality for sets which correspond to unit balls with respect to unconditional norms on \mathbb{C}^n . Some partial results concerning general case has been recently obtained in [Tko].

Definitions and preliminary statements are provided in Section 2. Section 3 is devoted to the main result. It also contains a proof of a one-dimensional inequality, which bounds entropy, and seems to be the heart of the proof of our main theorem.

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2 Preliminaries

We define the standard Gaussian measure ν_n on the space \mathbb{C}^n via the formula

$$\nu_n(A) = \gamma_{2n}(\tau(A)), \quad \text{for any Borel set } A \subset \mathbb{C}^n,$$

where $\mathbb{C}^n \xrightarrow{\tau} \mathbb{R}^{2n}$ is the bijection given by

$$\tau(z_1, \dots, z_n) = (\Re z_1, \Im z_1, \dots, \Re z_n, \Im z_n).$$

We adopt the notation $\mathbb{R}_+ = [0, +\infty)$. Later on we will also extensively use the notion of the *entropy* of a function $h: X \rightarrow \mathbb{R}_+$ with respect to a probability measure μ on a measurable space X

$$\text{Ent}_\mu f = \int_X f(x) \ln f(x) d\mu(x) - \left(\int_X f(x) d\mu(x) \right) \ln \left(\int_X f(x) d\mu(x) \right). \quad (1)$$

We say that a closed subset of \mathbb{C}^n *supports the complex S-inequality, SC-inequality* for short, if for any *cylinder* $C = \{z \in \mathbb{C}^n \mid |z_1| \leq R\}$ we have

$$\nu_n(K) = \nu_n(C) \implies \nu_n(tK) \geq \nu_n(tC), \quad \text{for } t \geq 1. \quad (2)$$

Note that the natural counterpart of *S-inequality* in the complex case is the following conjecture due to Prof. A. Pełczyński, which has already been discussed in [Tko].

Conjecture. *All closed subsets K of \mathbb{C}^n which are rotationally symmetric, that is $e^{i\theta}K = K$ for any $\theta \in \mathbb{R}$, support SC-inequality.*

In the present paper we are interested in the class \mathfrak{R} of all closed sets in \mathbb{C}^n which are *Reinhardt complete*, i.e. along with each point (z_1, \dots, z_n) such a set contains all points (w_1, \dots, w_n) for which $|w_k| \leq |z_k|$, $k = 1, \dots, n$ (consult for instance the textbook [Sh, I.1.2, pp. 8–9]). The key point is that this class contains all unit balls with respect to unconditional norms on \mathbb{C}^n . Recall that a norm $\|\cdot\|$ is said to be *unconditional* if $\|(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)\| = \|z\|$ for all $z \in \mathbb{C}^n$ and $\theta_1, \dots, \theta_n \in \mathbb{R}$.

The goal is to prove that all sets from the class \mathfrak{R} support SC-inequality. Now we establish some general yet simple observations which allows us to reduce the problem to a one-dimensional entropy inequality.

Proposition 1. *A closed subset K of \mathbb{C}^n supports SC-inequality if for any cylinder C we have*

$$\nu_n(K) = \nu_n(C) \implies \left. \frac{d}{dt} \nu_n(tK) \right|_{t=1} \geq \left. \frac{d}{dt} \nu_n(tC) \right|_{t=1}. \quad (3)$$

The proof is essentially given in [KS, Lemma 1], so we skip it. For any closed set A the derivative of the function $t \mapsto \nu_n(tA)$ is easy to compute. Indeed,

$$\begin{aligned} \left. \frac{d}{dt} \nu_n(tA) \right|_{t=1} &= \left. \frac{d}{dt} \int_{tA} e^{-|z|^2/2} dz \right|_{t=1} = \left. \frac{d}{dt} \int_A t^{2n} e^{-t^2|w|^2/2} dw \right|_{t=1} \\ &= 2n\nu_n(A) - \int_A |z|^2 d\nu_n(z). \end{aligned}$$

Moreover, the integral of $|z|^2$ over a cylinder C may be expressed explicitly in terms of the measure $\nu_n(C)$. Namely,

$$\int_C |z|^2 d\nu_n(z) = 2(1 - \nu_n(C)) \ln(1 - \nu_n(C)) + 2n\nu_n(C).$$

Combining these two remarks with the preceding proposition we obtain an equivalent formulation of the problem..

Proposition 2. *A closed subset K of \mathbb{C}^n supports SC-inequality if and only if*

$$\int_K |z|^2 d\nu_n(z) \leq 2n\nu_n(K) + 2(1 - \nu_n(K)) \ln(1 - \nu_n(K)). \quad (4)$$

3 Main result

We aim at proving the aforementioned main result, which reads as follows

Theorem 1. *Any set from the class \mathfrak{R} supports SC-inequality.*

We begin with a one-dimensional entropy inequality.

Lemma 1. *Let μ be a Borel probability measure on \mathbb{R}_+ and suppose $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded and non-decreasing function. Then*

$$\text{Ent}_\mu f \leq - \int_{\mathbb{R}_+} f(x) \left(1 + \ln \mu((x, \infty)) \right) d\mu(x). \quad (5)$$

Proof. Using homogeneity of both sides of (5), without loss of generality, we can assume that $\int_{\mathbb{R}_+} f d\mu = 1$. Then we may rewrite the assertion of the lemma as follows

$$\int_{\mathbb{R}_+} \ln \left(f(x) \int_{(x, \infty)} d\mu(t) \right) f(x) d\mu(x) \leq -1.$$

Introduce the probability measure ν on \mathbb{R}_+ with the density f with respect to μ . Thanks to monotonicity of f we might estimate the left hand side of the last inequality by

$$\int_{\mathbb{R}_+} \ln \left(\nu((x, \infty)) \right) d\nu(x) = - \int_0^\infty \int_0^1 \frac{du}{u} \mathbf{1}_{\{u \geq \nu((x, \infty))\}}(u, x) d\nu(x).$$

Define the function

$$H(y) := \inf \{t \mid \nu((t, \infty)) \leq y\},$$

which is the *inverse* tail function, and observe that

$$\{(u, x) \mid u \geq \nu((x, \infty))\} \supset \{(u, x) \mid H(u) \leq x\},$$

as $u \geq \nu((H(u), \infty)) \geq T(x)$. This leads to

$$\begin{aligned} - \int_0^\infty \int_0^1 \frac{du}{u} \mathbf{1}_{\{u \geq \nu((x, \infty))\}}(u, x) d\nu(x) &\leq - \int_0^\infty \int_0^1 \frac{du}{u} \mathbf{1}_{\{H(u) \leq x\}}(u, x) d\nu(x) \\ &= - \int_0^1 \nu([H(u), \infty)) \frac{du}{u}. \end{aligned}$$

Since $u \leq \nu([H(u), \infty))$, we finally get the desired estimation. \square

Now, for a certain class of functions, we establish the multidimensional version of inequality (5). For the simplicity, we formulate this result for the Gaussian measure.

Lemma 2. *Let $g: \mathbb{C}^n \rightarrow \mathbb{R}_+$ be a bounded function satisfying*

- 1) $g((e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)) = g(z)$ for any $z \in \mathbb{C}^n$ and $\theta_1, \dots, \theta_n \in \mathbb{R}$,
- 2) for any $w, z \in \mathbb{C}^n$ the condition $|w_k| \leq |z_k|$, $k = 1, \dots, n$ implies $g(w) \leq g(z)$.

Then

$$\text{Ent}_{\nu_n} g \leq \int_{\mathbb{C}^n} g(z) \left(\frac{|z|^2}{2} - n \right) d\nu_n(z). \quad (6)$$

Proof. One piece of notation: for a fixed vector $r = (r_1, \dots, r_n) \in (\mathbb{R}_+)^n$ we denote $r^k = (r_1, \dots, r_{k-1}, r_{k+1}, \dots, r_n) \in (\mathbb{R}_+)^{n-1}$, and then define the functions

$$g_k^{r^k}(x) = g(r_1, \dots, r_{k-1}, x, r_{k+1}, \dots, r_n), \quad k = 1, \dots, n.$$

Notice that for a function $h: \mathbb{C} \rightarrow \mathbb{R}_+$ obeying the property 1) we get

$$\int_{\mathbb{C}} h(z) d\nu_1(z) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty h(re^{i\theta}) e^{-r^2/2} r dr d\theta = \int_0^\infty h(r) d\mu(r),$$

where μ denotes the probability measure on \mathbb{R}_+ with the density at r given by $re^{-r^2/2}$. Therefore

$$\begin{aligned} \int_{\mathbb{C}^n} g(z) \left(\frac{|z|^2}{2} - n \right) d\nu_n(z) &= \int_{(\mathbb{R}_+)^n} g(r) \left(\frac{\sum_{k=1}^n r_k^2}{2} - n \right) d\mu^{\otimes n}(r) \\ &= \int_{(\mathbb{R}_+)^n} \sum_{k=1}^n \left[\int_{\mathbb{R}_+} g_j^{r^j}(x) \left(\frac{x^2}{2} - 1 \right) d\mu(x) \right] d\mu^{\otimes n}(r). \end{aligned}$$

Applying Lemma 1 for the function $g_j^{r^j}$ and the measure μ we obtain the estimation

$$\begin{aligned} \int_{\mathbb{C}^n} g(z) \left(\frac{|z|^2}{2} - n \right) d\nu_n(z) &\geq \int_{(\mathbb{R}_+)^n} \sum_{k=1}^n \text{Ent}_{\mu} g_j^{r^j} d\mu^{\otimes n}(r) \\ &\geq \text{Ent}_{\mu^{\otimes n}} g = \text{Ent}_{\nu_n} g, \end{aligned}$$

where the last inequality follows from subadditivity of entropy (for example see [Led, Proposition 5.6]). \square

Proof of Theorem 1. Fix $K \in \mathfrak{R}$. In order to show (4) we introduce the function $g(z) = 1 - \mathbf{1}_K(z)$. We adopt the standard convention that $0 \ln 0 = 0$, hence the desired inequality is equivalent to (6). Thus the application of Lemma 2 for the function g finishes the proof. \square

Theorem 1 immediately implies that the Cartesian products of cylinders support SC-inequality. As a consequence, SC-inequality possesses a tensorization property.

Corollary 1. *Assume sets $K_1 \subset \mathbb{C}^{n_1}, \dots, K_\ell \subset \mathbb{C}^{n_\ell}$ support SC-inequality. Then the set $K_1 \times \dots \times K_\ell$ also supports SC-inequality.*

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A S -inequality for the exponential measure in the unconditional case

Let λ be the symmetric exponential measure on \mathbb{R} , i.e.

$$d\lambda(x) = \frac{1}{2}e^{-|x|}dx, \quad x \in \mathbb{R},$$

and let $\lambda_n = \lambda \otimes \dots \otimes \lambda$ be the standard exponential measure on \mathbb{R}^n , i.e.

$$d\lambda(x) = \frac{1}{2^n}e^{-|x|_1}dx, \quad x \in \mathbb{R}^n,$$

where we denote $|(x_1, \dots, x_n)|_1 = \sum_{i=1}^n |x_i|$. It has been recently noticed that the technique of the paper applies also to the S -inequality for the measure λ_n . The result reads as follows

Theorem 2. *For any closed convex subset $K \subset \mathbb{R}^n$ which is unconditional, i.e. $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in K$ whenever $(x_1, \dots, x_n) \in K$ and $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$, and for any strip $P = \{x \in \mathbb{R}^n \mid |x_1| \leq p\}$, $p \geq 0$, we have*

$$\lambda_n(K) = \lambda_n(P) \implies \forall t \geq 1 \quad \lambda_n(tK) \geq \lambda_n(tP), \quad (7)$$

and, equivalently,

$$\lambda_n(K) = \lambda_n(P) \implies \forall t \leq 1 \quad \lambda_n(tK) \leq \lambda_n(tP). \quad (8)$$

Proof. The equivalence between (7) and (8) is straightforward. For instance, assume the latter does not hold. Then, there is $t_0 < 1$ such that $\lambda_n(t_0 K) > \lambda_n(t_0 P)$. So, we can find $s_0 < 1$ for which $\lambda_n(s_0 t_0 K) = \lambda_n(s_0 t_0 P)$. Using (7) we get a contradiction

$$\lambda_n(K) > \lambda_n(s_0 K) = \lambda_n\left(\frac{1}{t_0}(s_0 t_0 K)\right) \geq \lambda_n\left(\frac{1}{t_0}(t_0 P)\right) = \lambda_n(P) = \lambda_n(K).$$

We essentially follow the proof of Theorem 1. Hence, first of all, as in Proposition 1, we notice that we might equivalently prove that

$$\lambda_n(K) = \lambda_n(P) \implies \left. \frac{d}{dt} \lambda_n(tK) \right|_{t=1} \geq \left. \frac{d}{dt} \lambda_n(tP) \right|_{t=1}. \quad (9)$$

Again, a trivial computation shows that

$$\left. \frac{d}{dt} \lambda_n(tK) \right|_{t=1} = n\lambda_n(K) - \int_K |x|_1 d\lambda_n(x). \quad (10)$$

Thus, we would like to prove that

$$\lambda_n(K) = \lambda_n(P) \implies \int_K |x|_1 d\lambda_n(x) \leq \int_P |x|_1 d\lambda_n(x). \quad (11)$$

One another easy computation yields

$$\int_P |x|_1 d\lambda_n(x) = n(1 - e^{-p}) - pe^{-p}.$$

But $\lambda_n(K) = \lambda_n(P) = 1 - e^{-p}$, so we get

$$\int_P |x|_1 d\lambda_n(x) = n - n\lambda_n(K') + \lambda_n(K') \ln \lambda_n(K'),$$

where $K' = \mathbb{R}^n \setminus K$. Since $\int_{\mathbb{R}^n} |x|_1 d\lambda_n(x) = n$, (11) is equivalent to

$$-\lambda_n(K') \ln \lambda_n(K') \leq \int_{K'} (|x|_1 - n) d\lambda_n(x).$$

Now we introduce the function $g(x) = \mathbf{1}_{K'}(x)$. Then the above inequality may be rewritten to

$$\text{Ent}_{\lambda_n} g \leq \int_{\mathbb{R}^n} g(x)(|x|_1 - n) d\lambda_n(x). \quad (12)$$

Thanks to unconditionality of K the function g is even with respect to each coordinate. Using in addition convexity, we check that it is nondecreasing with respect to each coordinate. These properties as well as certain one-dimensional inequality, which is deduced from Lemma 1 for the measure with the density e^{-x} on \mathbb{R}_+ , allow us to prove inequality (12) in the same manner as in Lemma 2. \square

Following the method of [LO1, Corollary 3] we obtain the result concerning the comparison of moments.

Corollary 2. *Let $\|\cdot\|$ be a norm on \mathbb{R}^n which is unconditional, i.e.*

$$\|(\epsilon_1 x_1, \dots, \epsilon_n x_n)\| = \|(x_1, \dots, x_n)\|,$$

for any $x_j \in \mathbb{R}$ and $\epsilon_j \in \{-1, 1\}$. Then for $p \geq q > 0$

$$\left(\int_{\mathbb{R}^n} \|x\|^p d\lambda_n(x) \right)^{1/p} \leq C_{p,q} \left(\int_{\mathbb{R}^n} \|x\|^q d\lambda_n(x) \right)^{1/q}, \quad (13)$$

where the constant $C_{p,q} = \left(\int_{\mathbb{R}} |x|^p d\lambda(x) \right)^{1/p} / \left(\int_{\mathbb{R}} |x|^q d\lambda(x) \right)^{1/q}$ is the best possible.

Proof. The proof hinges on the fact that a ball $K = \{x \in \mathbb{R}^n \mid \|x\| \leq t\}$ with respect to the norm $\|\cdot\|$ is a closed convex unconditional set, so that Theorem 2 can be applied. \square

Remark 1. It has been tempting to see how the proof might have worked for product measures with the density $C_\rho^n e^{-\sum_{i=1}^n \rho(|x_i|)}$, where ρ is, e.g., positive convex and increasing function on $(0, \infty)$ and $\rho(0) = 0$. The only problem is that this is not entropy but another functional, constructed out of ρ , that appears in (12). In the case of functions $\rho(x) = x^p$, $p > 1$, we did check that Corollary 3 of [LO2] (consult there Definition 4 as well) does not give that such functionals possess desired properties such as subadditivity. From this point of view the exponential measure seems to be quite exceptional.

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